Stationary Solutions of Incompressible Stokes Equation for a Circular Pipe

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Stationary modes of the incompressible Stokes equation are derived using the method of potentials. Their relation to instationary modes is discussed.

In a recent paper [1] a method developed by Hansen, Stratton, Morse, Feshbach [2] ("HSMF-method") for solving linear vector-wave equations in electrodynamics was applied to find analytical expressions for three-dimensional time-dependent solutions of the Stokes equation for a pipe with circular cross-section. In this paper we derive analytical expressions for stationary modes of the stationary Stokes equation

$$\frac{1}{\varrho_0}\frac{\partial p}{\partial \boldsymbol{r}} = -\nu_0 \operatorname{rot}\operatorname{rot}\boldsymbol{u}\,, \quad \operatorname{div}\boldsymbol{u} = 0 \tag{1}$$

with the help of "potentials" a, b for the same geometry and boundary condition as in [1].

Expressing $\boldsymbol{u}$ and p by potentials $a(\boldsymbol{r}, t)$, $b(\boldsymbol{r}, t)$ ([1], ref. 24)

$$\boldsymbol{u} = \operatorname{rot}\boldsymbol{a} + \operatorname{rot}\operatorname{rot}\boldsymbol{b}\,, \quad p = p_0 + \varrho_0 \nu_0 \operatorname{div}(\boldsymbol{e}_z \Delta b)\,,$$
$$\boldsymbol{a} = a\,\boldsymbol{e}_z\,, \quad \boldsymbol{b} = b\,\boldsymbol{e}_z\,, \tag{2}$$

we get for a, b the differential equations

$$\Delta a = 0\,, \quad \Delta\Delta b = 0\,. \tag{3}$$

Introducing cylindrical coordinates r, φ, z and separating $a(r,\varphi,z)$ and $b(r,\varphi,z)$ into $a(r,\varphi,z) = a_r(r)\,a_\varphi(\varphi)\,a_z(z)$, $b(r,\varphi,z) = b_r(r)\,b_\varphi(\varphi)\,b_z(z)$ we find for $k \neq 0$ the solutions

$$a(r,\varphi,z) = a_m J_m(ikr)\, e^{im\varphi} e^{ikz}\,,$$
$$b(r,\varphi,z) = b_m J_m(ikr)\, e^{im\varphi} e^{ikz} + \frac{c_m r}{ik} J_m'(ikr)\, e^{im\varphi} e^{ikz}\,, \tag{4}$$

which give

$$u_r = \left[a_m \frac{im}{r} J_m + b_m k^2 \left(J_{m+1} - \frac{m}{ikr} J_m\right) + c_m \left(\frac{m^2}{ikr} - ikr\right) J_m\right] e^{im\varphi} e^{ikz}\,,$$
$$u_\varphi = \left[a_m ik \left(J_{m+1} - \frac{m}{ikr} J_m\right) - b_m \frac{mk}{r} J_m + c_m im \left(\frac{m}{ikr} J_m - J_{m+1}\right)\right] e^{im\varphi} e^{ikz}\,,$$
$$u_z = [-b_m k^2 J_m + c_m \{(2+m) J_m - ikrJ_{m+1}\}]\, e^{im\varphi} e^{ikz}\,. \tag{5}$$

The boundary condition $\boldsymbol{u}(r = R, \varphi, z) = \boldsymbol{0}$ leads to the dispersion relation

$$D_S(k,m) := ikRmJ_m^3(ikR) + [k^2R^2 - 2m(2+m)]J_m^2 J_{m+1} + ikR(2+3m) J_m J_{m+1}^2 + k^2R^2 J_{m+1}^3 = 0\,,$$
$$m = 0, 1, 2, \ldots\,. \tag{6}$$

For $m = 0$ we find

$$J_1 \left[J_0^2 + \frac{2i}{kR} J_0 J_1 + J_1^2\right] = 0\,, \tag{7}$$

which means either

a) $J_1(ik^{(j)}R) = 0\,, \quad k^{(j)} = ik_i^{(j)}\,,$
$(k_i^{(j)} > 0 \text{ for } z > 0)\,,$
$a_0 \neq 0\,, \quad b_0 = c_0 = 0\,, \quad j = 1, 2, 3, \ldots$ (7a)

or

b) $J_0^2 + \frac{2i}{k^{(j)}R} J_0 J_1 + J_1^2 = 0 \Leftrightarrow J_0 J_2 = J_1^2\,,$ (7b)
$a_0 = 0\,, \quad b_0 \neq 0\,, \quad c_0 \neq 0\,.$

See also [3], p. 547.

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Solutions $\boldsymbol{u}(r, \varphi, z)$ which belong to a) or b) are given by

a) $$\boldsymbol{u}^{(0,j)} = -\boldsymbol{e}_\varphi a_0 i k^{(j)} J_1(i k^{(j)} r)\, e^{i k^{(j)} z} = -\boldsymbol{e}_\varphi \tilde{a}_0^{(j)} J_1(k_i^{(j)} r)\, e^{-k_i^{(j)} z}, \quad \tilde{a}_0^{(j)} = a_0 k_i^{(j)}, \tag{8a}$$

b) $$u_r^{(0,j)} = \tilde{b}_0^{(j)} \left[J_1(i k^{(j)} r) - \frac{r}{R} \frac{J_0(i k^{(j)} r)}{J_0(i k^{(j)} R)} J_1(i k^{(j)} R) \right] e^{i k^{(j)} z}, \quad u_\varphi^{(0,j)} = 0, \tag{8b}$$

$$u_z^{(0,j)} = \tilde{b}_0^{(j)} \left[\frac{\{2 J_0(i k^{(j)} r) - i k^{(j)} r J_1(i k^{(j)} r)\} J_0(i k^{(j)} R)}{2 J_0(i k^{(j)} R) - i k^{(j)} R J_1(i k^{(j)} R)} - J_0(i k^{(j)} r) \right] e^{i k^{(j)} z},$$

$$a_0 = 0, \quad c_0 = \frac{b_0 (k^j)^2 J_1(i k^{(j)} R)}{i k^{(j)} R J_0(i k^{(j)} R)}.$$

Solutions for $m \neq 0$ can be found from (5) with

$$c_m^{(j)} = \tilde{b}_m^{(j)} \frac{J_m(i k^{(j)} R)}{(2+m) J_m(i k^{(j)} R) - i k^{(j)} R J_{m+1}(i k^{(j)} R)},$$

$$a_m^{(j)} = \tilde{b}_m^{(j)} \frac{i R}{m J_m(i k^{(j)} R)} \frac{\left(J_{m+1} - \frac{m}{i k^{(j)} R} J_m\right) \left[(2+m) J_m - i k^{(j)} R J_{m+1}\right] + \left(\frac{m^2}{i k^{(j)} R} - i k^{(j)} R\right) J_m^2}{(2+m) J_m(i k^{(j)} R) - i k^{(j)} R J_{m+1}(i k^{(j)} R)}, \tag{9}$$

where $k^{(j)}$ are solutions of (6) and $\tilde{b}_m^{(j)} = b_m (k^{(j)})^2$. The stationary pressure functions $p^{(m,j)}(r, \varphi, z)$ for $k^{(j)} \neq 0$, which follow from (2), (4) are given by

$$p^{(m,j)}(r, \varphi, z) = p_0 - 2 \varrho_0 \nu_0 i k^{(j)} c_m^{(j)} \cdot J_m(i k^{(j)} r)\, e^{i m \varphi} e^{i k^{(j)} z}. \tag{10}$$

If $k^{(j)}$ is a solution of (6) the same holds for $-k^{(j)}$. Numerical calculations for $m = 0$ ([1]) with small but finite values of σ are in agreement with (7a) to (8b). As can be seen from (6), there exist no solutions for $k^{(j)}$ with

$$J_m'(i k^{(j)} R) = -J_{m+1}(i k^{(j)} R) + \frac{m}{i k^{(j)} R} J_m(i k^{(j)} R) = 0, \quad m > 0,$$

as might be supposed from Figs. 1, 2 in [1].

The analytical limit process $\sigma \to 0$ applied to (19) of [1], where σ is the time-separation constant, does not give the dispersion relation for the stationary modes, since $\lim_{\sigma \to 0} D(k, m, \sigma) \equiv 0$. $D(k, m, \sigma)$ denotes the right hand side of equation (19) in [1] for incompressible flow. Instead, the following relation holds ($k \neq 0$):

$$D_S(k, m) = \text{const} \left. \frac{\partial D(k, m, \sigma)}{\partial \sigma} \right|_{\sigma = 0}. \tag{11}$$

As can be shown by simple but lengthy calculations a similar relation can be found between the stationary modes $\boldsymbol{u}^{(m,j)}(r, \varphi, z)$ and the instationary solutions $\boldsymbol{u}^{(m)}(r, \varphi, z, t, k, \sigma)$ (equation (17) in [1]) which fulfill the boundary condition $\boldsymbol{u}^{(m)}(r = R, \varphi, z, t, k, \sigma) = \boldsymbol{0}$:

$$\boldsymbol{u}^{(m,j)}(r, \varphi, z) = \left. \frac{\partial [\boldsymbol{u}^{(m)}_{(r, \varphi, z, t, k, \sigma)} e^{\sigma t}]}{\partial \sigma} \right|_{\substack{\sigma = 0 \\ k = k^{(j)}}} \cdot A(m, k^{(j)}) \quad (a_0 = 0 \text{ for } m = 0). \tag{12}$$

$A(m, k^{(j)})$ are constants and $k^{(j)}$ is a solution of (6). The only exception of relation (12) is the stationary solution for $m = 0$, $a_0 \neq 0$, $b_0 = c_0 = 0$. In this special case $\boldsymbol{u}^{(0,j)}(r, z)$ follows directly from

$$\boldsymbol{u}^{(0)}(r, z, t, k, \sigma) = -\boldsymbol{e}_\varphi a_0 \sqrt{\frac{\sigma}{\nu_0} - k^2}\, J_0'\left(\sqrt{\frac{\sigma}{\nu_0} - k^2}\right) e^{i k z} e^{-\sigma t} \tag{13}$$

for $\sigma = 0$ since $\boldsymbol{u}^{(0)}(r, z, t, k, \sigma = 0) \neq \boldsymbol{0}$. For $\sigma = 0$ ($a_0 = 0$ for $m = 0$) one gets

$$\lim_{\sigma \to 0} \boldsymbol{u}^{(m)}(r, \varphi, z, t, k, \sigma) = \boldsymbol{0}.$$

The relation between the stationary pressure functions (10) and the corresponding instationary ones

(Eq. (16) of [1] for $c_c = \infty$) is found to be

$$p^{(m,j)}(r, \varphi, z) = p_0 + \left.\frac{\partial [p^{(m)}_{(r,\varphi,z,t;k,\sigma)} e^{\sigma t}]}{\partial \sigma}\right|_{\substack{\sigma=0 \\ k=k^{(j)}}} \cdot B(m, k^{(j)}) , \quad (14)$$

where $B(m, k^{(j)})$ are constants. Without explicit knowledge of the stationary functions $D_S(k, m)$, $\boldsymbol{u}^{(m,j)}(r, \varphi, z)$, $p^{(m,j)}(r, \varphi, z)$ it does not seem possible to state the relations (11), (12), (14). In order to get the relations (12) and (14), the process $\sigma \to 0$ with $b_m \sigma = b_m^* = \text{const}$ must be used, where the constants b_m enter the functions $\boldsymbol{u}^{(m)}$, $p^{(m)}$ according to equation (18) of [1] for $c_c = \infty$. If one interprets the functions $\boldsymbol{u}^{(m)}$ or $\boldsymbol{u}^{(m,j)}$ for $k \neq 0$ as flow patterns in the pipe, the corresponding Reynolds-numbers must be assumed to be very small, which means that the coefficients a_0, b_m must be sufficiently small.

The case of vanishing separation constant k is of certain interest. If one solves (1) to (3) for $k = 0$ and $\boldsymbol{u}(r = R, \varphi, z) = \boldsymbol{0}$, only the trivial solution $\boldsymbol{u} \equiv \boldsymbol{0}$, $p = \text{const.}$ results: especially the Hagen-Poiseuille-solution for $m = 0$ does not appear. The reason is that the Hagen-Poiseuille-solution $\boldsymbol{u}_{\text{HP}}(r) = u_0(1 - r^2/R^2)\,\boldsymbol{e}_z$ cannot be represented by (3). For $k = 0$, $m = 0$, however, (1) is solved by

$$\boldsymbol{u}^{(0,0)} = \text{rot} \cdot \text{rot}\, \boldsymbol{b}_r^{(0,0)}(r) , \quad p^{(0,0)} = p_0 - c\,\varrho_0 \nu_0 z ,$$
$$\boldsymbol{b}_r^{(0,0)} = \boldsymbol{e}_z b_r^{(0,0)}(r) , \quad \Delta_r \Delta_r b_r^{(0,0)} = c ,$$
$$\Delta_r = \frac{1}{r}\left(r \frac{\mathrm{d}}{\mathrm{d}r}\right) , \quad (15)$$

where c is an arbitrary constant. The solution of $\Delta_r \Delta_r b_r^{(0,0)}(r) = c$ consists of the general solution of the homogeneous equation and a special solution of the inhomogeneous equation. Taking into account the non-trivial, non-singular terms only, we have

$$b_r^{(0,0)}(r) = b_0 r^2 + \frac{c}{64} r^4 + b_1 r^2 \ln r ,$$
$$\Delta b_r^{(0,0)} = \frac{c\,r^2}{4} + 4 b_0 + 4 b_1 (\ln r + 1) , \quad b_1 = 0 , \quad (16)$$

which gives

$$\boldsymbol{u}^{(0,0)}_{(z)} = \frac{c}{4R^2}\left(1 - \frac{r^2}{R^2}\right) \boldsymbol{e}_z , \quad p^{(0,0)}_{(z)} = p_0 - c\,\varrho_0 \nu_0 z . \quad (17)$$

As already Sexl [3] observed, the Hagen-Poiseuille solution can be found also from the time-dependent solutions of the Stokes equation for $m = 0$, $k = 0$ by the process $\lim_{\substack{\sigma \to 0 \\ c\sigma = \text{const}}} \boldsymbol{u}_S^{(0)}(r, t, \sigma)$, where $\boldsymbol{u}_S^{(0)}$ denotes the functions found by Sexl ([3], p. 575). In order to find all stationary solutions for $k = 0$ from the time-dependent solutions of the Stokes equation by using the HSMF-method we take the solutions found by Brosa [1]. Since he did not separate the Helmholtz-equation for $k = 0$ quite correctly, we summarize the three types of time-dependent solutions for $k = 0$:

a) $m = 0$, α_0 arbitrary, b_0 arbitrary:

$$J_1\left(\sqrt{\frac{\sigma_1^{(j)}}{\nu_0}}\, R\right) = 0 , \quad j = 1, 2, 3, \ldots , \quad u_r^{(0)} = 0 ,$$

$$u_\varphi^{(0)} = a_0 \sqrt{\frac{\sigma_1^{(j)}}{\nu_0}}\, J_1\left(\sqrt{\frac{\sigma_1^{(j)}}{\nu_0}}\, r\right) e^{-\sigma_1^{(j)} t} ,$$

$$u_z^{(0)} = b_0 \frac{\sigma_1^{(j)}}{\nu_0}\left[J_0\left(\sqrt{\frac{\sigma_1^{(j)}}{\nu_0}}\, r\right) - J_0\left(\sqrt{\frac{\sigma_1^{(j)}}{\nu_0}}\, R\right)\right] e^{-\sigma_1^{(j)} t} . \quad (18)$$

b) $m = 0$, b_0 arbitrary, σ arbitrary:

$$u_r^{(0)} = 0 , \quad u_\varphi^{(0)} = 0 ,$$

$$u_z^{(0)} = b_0 \frac{\sigma}{\nu_0}\left[J_0\left(\sqrt{\frac{\sigma}{\nu_0}}\, r\right) - J_0\left(\sqrt{\frac{\sigma}{\nu_0}}\, R\right)\right] e^{-\sigma t} . \quad (19)$$

c) $m = 0, 1, 2, \ldots$, b_m arbitrary:

$$J_m\left(\sqrt{\frac{\sigma_m^{(j)}}{\nu_0}}\, R\right) = 0 , \quad u_r^{(m)} = 0 , \quad u_\varphi^{(m)} = 0 ,$$

$$u_z^{(m)} = b_m \frac{\sigma_m^{(j)}}{\nu_0} J_m\left(\sqrt{\frac{\sigma_m^{(j)}}{\nu_0}}\, r\right) e^{i m \varphi} e^{-\sigma_m^{(j)} t} . \quad (20)$$

If we look for stationary solutions with $k = 0$ as result of a process $\sigma \to 0$, $b_m \sigma^n = \text{const}$, only solutions of type b) can be used since in the other two cases $\sigma = 0$ cannot be approached smoothly. Making explicit the argument used in [1] we get for

$$u_z^{(0)}(r, t, \sigma) = b_0 \frac{\sigma}{\nu_0}\left[J_0\left(\sqrt{\frac{\sigma}{\nu_0}}\, r\right) - J_0\left(\sqrt{\frac{\sigma}{\nu_0}}\, R\right)\right] e^{-\sigma t}$$

the relations

$$u_z^{(0)}(r, t, \sigma = 0) = 0\,, \quad \left(\frac{\partial u_z^{(0)}(\sigma)}{\partial \sigma}\right)_{\sigma=0} = 0\,,$$

$$\left(\frac{\partial^2 u_z^{(0)}(\sigma)}{\partial \sigma^2}\right)_{\sigma=0} = \frac{b_0}{2 v_0^2}(R^2 - r^2)\,.$$

Thus we find

$$\lim_{\substack{\sigma \to 0 \\ b_0 \sigma^2 = b_0^*}} \boldsymbol{u}^{(0)}(r, t, \sigma) = \frac{b_0^*}{4 v_0^2}(R^2 - r^2)\, \boldsymbol{e}_z = u_0\left(1 - \frac{r^2}{R^2}\right) \boldsymbol{e}_z =: \boldsymbol{u}_{\mathrm{HP}}(r)\,,$$

$$u_0 = b_0^* R^2 / 4 v_0^2\,.$$

In the same way the stationary pressure $p_{\mathrm{HP}}(z)$ follows from

$$p_{\mathrm{HP}}(z) = \lim_{\substack{\sigma \to 0 \\ b_0 \sigma^2 = b_0^*}} p^{(0)}(z, t, \sigma)\,,$$

where $p^{(0)}(z, t, \sigma)$ is given by ([1], Eq. (27))

$$p^{(0)}(z, t, \sigma) = p_0 - \varrho_0 b_0 \frac{\sigma^2}{v_0} J_0\left(\sqrt{\frac{\sigma}{v_0}}\, R\right) z\, \mathrm{e}^{-\sigma t}\,.$$

With

$$p^{(0)}(z, t, \sigma = 0) = p_0\,, \quad \left(\frac{\partial p^{(0)}(\sigma)}{\partial \sigma}\right)_{\sigma=0} = 0\,,$$

$$\left(\frac{\partial^2 p^{(0)}(\sigma)}{\partial \sigma^2}\right)_{\sigma=0} = -\frac{2 \varrho_0 b_0}{v_0} z$$

we get finally

$$p_{\mathrm{HP}}(z) = p_0 - \frac{\varrho_0}{v_0} b_0^* z = p_0 - 4 \varrho_0 v_0 \frac{u_0}{R^2} z\,.$$

Contrary to the stationary solutions $\boldsymbol{u}^{(m,j)}(r, \varphi, z)$, the coefficient b_0^* (or u_0) which appears in $\boldsymbol{u}_{\mathrm{HP}}(r)$ may take values which give large Reynolds-numbers since $\boldsymbol{u}_{\mathrm{HP}}(r)$ solves the stationary non-linear Navier-Stokes equation.

With regard to the infinite pipe the Hagen-Poiseuille-solution remains as the only stationary solution of the Stokes equation which fulfills the (incomplete) boundary condition $\boldsymbol{u}(r = R) = \boldsymbol{0}$, which possesses continuous partial derivatives everywhere and which is bounded for $-\infty < z < \infty$. The functions $\boldsymbol{u}^{(m,j)}(r, \varphi, z)$ with $k \neq 0$ represent solutions of the Stokes equation for the semi-infinite pipe with $\boldsymbol{u}^{(m,j)}(r, \varphi, z \to \infty) = \boldsymbol{0}$ and a prescribed boundary condition at $z = 0$.

We want to mention that also $\Delta a = 0$ (2) is only sufficient to solve (1). If for the case $m = 0$, $k = 0$ one lets $\Delta a = c$, one finds an additional solution $\boldsymbol{u}(r) = -\frac{1}{2} c\, r\, \boldsymbol{e}_\varphi$, which gives no contribution for homogeneous boundary conditions $\boldsymbol{u}^{(0,0)}(r) = \boldsymbol{0}$, but for a boundary condition

$$u_r^{(0,0)}(r = R) = u_z^{(0,0)}(r = R) = 0\,,$$

$$u_\varphi^{(0,0)}(r = R) = u_{\varphi 0}\,,$$

which characterizes a rotating pipe.

[1] U. Brosa, Z. Naturforsch. **41 a,** 1141 (1986).
[2] J. A. Stratton, Electromagnetic Theory, McGraw-Hill New York 1941. P. M. Morse and H. Feshbach, Methods of Theoretical Physics, Part II, McGraw-Hill, New York 1953.
[3] Th. Sexl, Ann. Phys. **87,** 570 (1928).